On Mathematics Education: the Lakatosian Revolution*

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When a philosopher like me is invited to address a professional group like the present audience, it is unreasonable to expect that he show expertise in their specialization of the level required from one addressing a conference of his peers in his own specialization. Hence, such an invitation must be based on a different expectation. Possibly the philosopher is expected to perform a ritual function, akin to that of a priest. I have been invited for ritual purposes many times in the past, but no more. One advantage of a reputation is that it prevents such understandable cross-purposes. Whatever the reason for my being invited, it is no longer to offer platitudes or homilies. I will neither soothe nor preach, and by now this is known. An invitation to an outsider may also be an invitation to impart to one field the fruits of another. For example, mathematicians were invited to physics conferences to teach professional physicists some Lie algebra. I do have some specialized results to tell you about, yet hardly from the field of philosophy. But there is one good reason for inviting a philosopher to any specialized conference: he may be able to make quite a lot of trouble in a short time.

I am telling you this in advance because of my past experiences. There is no need to accept my offer to make trouble, and like everyone else I can be dismissed on any one of many sorts of pretext. I often meet indignation, and I have one thing to say to the indignant: his indignation is but an excuse for dismissing me, and dismiss me he may anyway, so that his indignation is redundant. And, as Spinoza noticed, it is expensive. So it is better to dismiss me with no indignation if dismiss me you will.

The indignant, if I may pursue the matter for one more paragraph, will now vehemently protest, though vehemence is, like indignation, an expensive redundancy. He will say that he is indignant not because of his own personal reasons but because I am dangerous to the system. I appreciate the compliment and for a fleeting megalomaniacal moment I am even tempted to accept it. But now is my lucky break and perhaps you and I together may become a little dangerous. As I say, I have come here to try and stir up a little trouble.

We are all working within a system, both in a broad sense of the word and in a narrow sense. And my first thesis to you, which is my first import — from philosophy this time — is that one's efficiency much depends on the answer to the question, in which framework is one operating? To give you a few examples. In mathematics the same theorem may be trivial in one system and deep in another. At times it is worthwhile to embed systems within systems so as to make an easy kill and import some results. Abraham Robinson, for example, proved that all theorems of analysis in a non-standard interpretation are valid in the standard interpretation and that some theorems which are deep in the standard reading are trivial in the non-standard one. A corollary can be derived from this for mathematics education. If, instead of moving along in a lecture course in mathematics as fast as possible, proving all the necessary theorems, introducing all the necessary tools, etc., if instead of this one takes it slowly and explains to one's audience what one is doing and one's rationale for it, then one's audience will find the course both more fun and more rewarding. I will return to this later.

Another example. The problem of induction is easily soluble in one system and utterly insoluble in another. I will not enter this vast topic now but only mention results of other studies of mine. When one assumes Daltonian chemistry, one can employ induction when performing complex tasks of chemical analysis. The system is known to be false, since some atoms are unstable and disintegrate into new atoms, contrary to the basic assumptions of Daltonian chemical analysis, yet in many cases this is irrelevant. If, however, one clings to the classical view of science as empirically demonstrable and hence error-free, then the solution here outlined must be rejected as faulty, or at least as question-begging. Similarly in technology. An airline which follows legal requirements is not necessarily to be condemned when a vessel which belongs to it is disastrously destroyed. It can claim that it has complied with the law and thus has adequately discharged its responsibility. The government agency operates within a framework too, and this may justify its having unknowingly permitted the faulty vessel to fly. Of course, both the airline company and the government agency, perhaps also other parties, may be found reprehensible, but the catastrophe alone is not proof of neglect. The solution to the problem of induction, if perfect, would allow one to preclude all the imperfections that permit faulty vessels to be used, and would make such use, then, automatically reprehensible.

My final example is Marxism. Not the one officially endorsed by communist governments at home, which most of us are quite unfamiliar with, but in the West. Of all the vaguenesses of western Marxist thinking, I pick up the one about the framework: which framework do Marxists wish to destroy? We can see the vagueness of most Marxists by observing the clarity of those who are still clear. For example, Ivan Illich wants the whole social system destroyed: he thinks society will be better off for that; perhaps he also thinks that once the current system be destroyed, a better one will emerge by itself. If so, he does not tell us how. But he quite rightly observes that the school system is a pillar of the social system — this is true of all social systems — and so he wants it destroyed. How, then, will the education of the members of society proceed after the revolution? Will a

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new school system emerge all by itself? No, says Ivan Illich. We do not need one, he avers. Following Paul Goodman he claims that children in the park can learn more from the elderly folk who sit on the benches to warm their bones than from professional teachers. I do not wish to deny that this vision holds for some cases. Nor am I totally insensitive to its beauty and allure. Yet I do think that as a substitute for the present system it is simply disastrous. It will destroy the present system, for sure; but it will destroy everything else as well.

Hence most Marxist thinkers are muddled because they do not want to share Illich's framework but cannot provide an alternative that is at all convincing. Other people respond to Illich either by endorsing his vision or by going ultra conservative. Both responses are conservative in practice since Illich's alternative is plainly a non-starter.

I do not mean to condemn Illich for his extremism but to congratulate him on his relative clarity. As to extremism, his is by no means the limit. For example, the idea of the discovery method, though in one framework highly conservative — i.e. in that it keeps the classroom structure as it is — in another framework is more extremist and radical than Goodman or Illich. It says that by sheer prompting and coaxing a child can discover for himself what the greatest minds in the history of mathematics did over twenty-five centuries. This is a megalomaniacal view either of prompting and coaxing *per se* or of the prompting and coaxing teacher.

So let me describe now the exact framework or system that I wish to see destroyed, the one I wish to see replace it, and the way I propose to effect such a revolution: the Lakatosian revolution, as I have elsewhere called it. And I begin with what I have already covered in this brief introduction. First, I do not wish to see schools closed down or teachers dismissed. Second, I oppose prompting, coaxing, motivation, and any other forms of leadership. Instead I recommend that teachers, like any other adult citizens, explain to their audience their purposes and invite them to partake in exciting intellectual and other ventures.

Let me start, then, with the last point, with motivation versus declaring one's purpose. There are two systems within which the matter just introduced may be discussed. First, educational coaxing and motivation is a form of deception. To take an example from Moses Maimonides, a teacher offers fruits and nuts to children as a reward for their acquisition of literacy. The children, says Maimonides, may think they work for peanuts. And, as the teacher has promised these, he must indeed reward his pupils with peanuts. But the true reward lies elsewhere. The parent says to his child, the teacher says to his pupil, when you are adult you will comprehend my purpose and see its benevolence towards you and then you will thank me. Motivation, then, is making the child do what he has no inclination to do. One can achieve this by offering him peanuts, by offering him the absence of flogging (that is to say, by promising him flogging unless he is motivated; and then, is one still obliged to flog the lazy pupil?), and, worst of all, one can achieve this by promising love to the good pupil and no love for the lazy one. Replacing the rationing of love for the rationing of non-harassment is one of the most famous, most highly praised advances of the modern age, of the age of enlightened education. One of the clearest results from the field of criminology and of psychiatry is that the rationing of love, regardless of any other factor that may or may not accompany it, the sheer system of rationing of love, is more damaging and has more lasting effects than flogging or threats to flog. I need not mention the obvious fact that the discovery method has been tried chiefly in experimental and/or progressive schools where enlightened teachers use every subtle psychological means, except brute physical violence, to make their pupils work their hardest to achieve the impossible. My horror of this knows no bounds: I find physical violence merciful in comparison to sophisticated psychological manipulation.

The claim that stands behind this manipulation, may I repeat, is that a child is unable to comprehend the true purpose of the exercise. And this enables one to switch from the educational framework or system to the political framework or system. We can ask, what is the purpose of education? It may be to conserve; it may be to raise the cadres for the revolution; it may be to help our successors do better than we have done. To cut a long story short, I endorse the third purpose. Education, then, is for independent, free, able citizens. Hence I am for the true discovery method, not for the method of coaxing and cajoling and manipulating children, but of talking to them as to equals and of discussing with them the past faults of the system, the joys of discovery and of improvements and of innovation, and their desire to evolve their necessary faculties.

The claim of the establishment still precludes this. The establishment, to repeat yet again, denies that young ones know or can know the purpose of the exercise. We thus have to shift our framework yet again, from the psychology of education, through the politics of education, to the intellectual framework that the educator himself endorses, and not *only* as an educator, but also as a citizen in the broader sense. He claims authority over his charge on the pretext that his charge does not know. Before entering on a discussion of this claim, let us notice the implied claim that *he* does. Does he?

I am told that some of you are not familiar with the great classic work of Imre Lakatos, *Proofs and Refutations*. I much recommend it as the most revolutionary work in the philosophy of mathematics in the post-World-War-II period, no less profound and wider in scope than Paul Cohen's discovery or the discovery of category theory or anything else. What Lakatos has succeeded in showing is that all mathematical presentations in standard mathematics textbooks are ill-conceived as to their purpose, and that mathematicians, even some of the greatest, are surprisingly vague about the rationale of mathematics, and about the aims and methods of mathematical proofs in particular.

The framework within which these people operate is what handicaps them, and to that extent it makes their success as great mathematicians all the more miraculous. I canot discuss this framework here: I have devoted a whole book to it. But I can describe it briefly. The framework, devised by Parmenides and developed by Plato, is of particular significance for the understanding of the history of Greek mathematics, and thus of the history of mathematics at large, as was ably argued by Árpád Szábó, the person who was Lakatos' teacher in their native Hungary.

Parmenides divided our whole cultural and intellectual and moral world into two extreme poles, truth and falsity. Truth is demonstrable and gives us the nature of things. Falsity is mere appearance and convention. To be more specific at the cost of adding to the original text, Parmenides and Plato were not interested in any old falsehood, in blunt mathematical error, or in cock-and-bull stories and fairytales. Rather, they were disturbed by prevalent error, what is at times called truth by convention or by agreement or by popular support. There is truth by convention which is mere *doxa*, i.e. opinion, i.e. popular opinion, which is mere appearance, i.e. seemingly convincing. And there is truth by reason, i.e. by proof, i.e. *logon piston* or *episteme*, i.e. knowledge proper. In the whole field of western thought this dichotomy runs through. In bullish mood thinkers opted for knowledge, in bearish mood they settled for convention. Even in bullish mood, the status of truth by convention was granted as a consolidation prize. Thus, Maxwell gave his own theory first prize and that of Lorenz (the Dane) the consolation prize. Some political philosophers, Edmund Burke and Hegel in particular, tried to defend their reactionary philosophy by appeal to both nature and convention. Yet generally most naturalists are radicalists — like Ivan Illich — and most conventionalists are conservative — like James Bryant Conant who defended conventionalism in science: scientific truth, he said, in his celebrated Harvard Case Studies and elsewhere, is truth by convention.

When we come to any branch of learning, but particularly to logic and mathematics, the dichotomy of nature and convention is so dreadful because it cuts out purpose: nature leaves no room for my desires and convention makes them arbitrary. This is why in logic and in mathematics most philosophers are either naturalists, logicists, ideal language theorists, etc., or formalists who deem any axiom system as good as any other. Both are in error: systems are man-made but not arbitrary; they are designed to answer certain desiderata, and these desiderata are themselves subject to debate.

I now come to a major fork. The structure of my discussion moves here. My desideratum, I have told you, is to overturn the educational system. I have delineated my framework: we want to decide what are the desiderata of mathematics education — or education in general — and show the system wanting with respect to them, devise a better system, and transact the change. In particular, I have said, mathematics teachers do not quite know the aim of mathematics education, as Lakatos has proven, and the philosophies of both mathematics and education are cast in a Parmenidean-Platonic framework that tones down desiderata, so we have a big task on our hand which we have hardly begun. We can now move (1) to the purpose of education, with special reference to mathematics; or (2) to the purpose of mathematical instruction with special reference to the education of mathematics teachers, of mathematics researchers, applied mathematicians, and amateurs; and we can discuss (3) the aim of mathematical research.

I do not know the aim of mathematical research. I could say "to discover the nature of mathematics", and I could say "to devise the mathematical system most useful for our studies of nature or for our conquest of nature". Doing so casts mathematics well within the Parmenidean-Platonic framework. It is the enormous merit of Lakatos that he has trussed this framework's seams. Even Szabó and Polya did not touch on this issue. Karl Popper, Lakatos' other teacher, even if not a Hungarian, has devised a philosophy of science and a social philosophy that do not fit the Parmenidean-Platonic dichotomy between nature and convention; yet in logic and mathematics he was a conventionalist until he met Lakatos. Lakatos himself was no less shaken by Popper than Popper by Lakatos. Popper said of science, and Lakatos said of mathematics, that each is full of errors to be corrected and hence is neither nature, i.e. not only demonstrable final truths, nor convention, i.e. not truths arbitrarily nailed down and defended against criticism. They agreed that we hope to progress towards the truth, or nature. At times, though, Lakatos seems to suggest that we progress towards meeting conceptual desiderata. Lakatos wanted to say the same of logic. But he was persuaded that, as both Popper and Quine argue, the law of contradiction has a special status: we cannot hope to criticize it; for to criticize it is to discover a true contradiction; which is absurd. Lakatos, then, was swayed by Popper to become a conventionalist in logic and Popper was swayed by Lakatos to reject conventionalism in mathematics. The result is that neither of them offered a comprehensive view.

This is not a critique. It is clear that mathematics is more deeply linked with language than physics. I have no need to mention the contribution of formalization to mathematics, from Hilbert to Cohen. No attempt to formalize physics has done anything for physics, as yet — to philosophy and to mathematics, but not to physics. Moreover, whereas with Hilbert and Gödel and von Neumann it is not hard to declare each of their papers as belonging essentially to logic *or* to mathematics, model theory, at least since Abraham Robinson, is decidedly both a central contribution to logic *and* to mathematics, as Gödel noticed in his comment on Robinson.

In view of all this, I hope I am allowed to conclude that the field is in a fluid state and be forgiven for my ignorance. We once knew at least what objects mathematics handled, whether numbers and figures, or groups and fields. With the stalemate in the foundation of mathematics, one can conclude that even the question of what mathematics is about is very much an open question.

This leads me to training. Two major evils are specific to training if we leave to the end what is wrong with education. Ever since Euclid became a standard text, and at least since the days when Archimedes wrote his breathtaking masterpieces, it has been customary, more so in mathematics than in any other field of study, to train through the teaching of textbooks. Textbooks have characteristics that have evolved, but which started as the characteristics of Euclid's *Elements*. Anyone in doubt should look at any geometry textbook and see how much it is indebted to either Euclid or Hilbert, not to mention Hilbert's conscious debt to Euclid. Historians are now arguing about Euclid's aims: why did he write his classic book? I do not know. I cited a famous sentence of Proclus, in which he says Euclid was executing Plato's program, in the presence of some leading historians of mathematics, and one of them, a very clever fellow, said to me: I like Proclus' statement since I am biased in favour of metaphysics, otherwise I could argue against it just as well. He said that Proclus was not the most reliable author in antiquity. I conceded. This exchange took place after a dramatic public exchange between Szabó and Jaakko Hintikka about the question: what was Euclid's aim? Why did he not tell us? Clearly he did not know we would be so interested.

The Lakatos revolution is the end of the textbook. Hilbert told us why he wrote his book on geometry, and was generally excellently clear about his aims. There is still a debate about his program, but at least he had one. The study of mathematics in the future will be frankly programmatic: programs will be put on the agenda, everyone will belong to the steering committee that will decide the agenda, simply because groups with different agenda will do different things and they will then pool information and rediscuss the agenda. I am predicting the future on the basis of a hypothesis that learning by agenda is ever so much more powerful than learning by textbook, that by natural or competitive selection the one to introduce it will be the winner.

And the increased efficiency is two-fold. First, specificity. The mathematics required by (1) the amateur, (2) the applied mathematician, (3) the mathematics teacher, and (4) the research mathematician, are so very different that each needs a different agenda. Even when all four want to know what an axiom is, each of them will doubtless approach matters differently. And agenda-making is active student-participation, and educational psychology is unequivocal about certain matters: there is no better training than by active participation, trial and error, etc. Here a few writers in different fields collude: Wiener in cybernetics, Piaget in developmental psychology, Chomsky in psycholinguistics, Popper in scientific method; they all favour active participation. So I can move on without driving home this point any further.

I wish to speak of education proper now. The worst thing about motivation theory, to return to my old *bête noire*, is not that it is an advocacy of lies, though it is; not that it is based on contempt for the one to be educated, an unjust expression of superiority of the educator, though it is that too; the worst of it is that it is a system of training for dependence. The teacher makes a simple mistake, has a simple optical illusion. He wants his student to listen to him, and for that he demands obedience to himself. When he does not get it, and children are hardly ever fully obedient by the clock, he feels justified in breaking their backs. He then breeds well-read, well-trained professionals lacking any backbones.

I wish to state my view, if I may, that even the most intellectual, most abstract achievement, is impossible without a measure of moral independence, since without it the Church would still dominate the universities today as when it founded them. I also wish to state, if I may, that independence is a way of life, and a constant struggle. Allow me to illustrate this last point. I have always deemed myself fairly independent, but then I noticed that once, after I had read a paper and received an adverse comment, I did not handle the comment as maturely as I should have done. It was a few years ago; both Professor Grünbaum and I were guests together in some German university, and he did me the honour of attending my lecture. He said it was ill-prepared and half-digested and advised me to put it aside. I did so. After a couple of years I read it and decided I had been browbeaten. I do not wish to blame friendly Professor Grünbaum, nor to express a conviction that my paper was good and his judgment incorrect: for all I know he was right. Rather, I want to confess I was easily browbeaten, and though he only advised me I took it as a browbeating and capitulated instead of attempting to form a judgment of my own — whether in agreement with his or not. There was no harm done, and the paper will soon be printed, I hope. But I wanted to indicate how easy it is to be browbeaten. How much easier it was to be browbeaten when I was an inexperienced, bewildered pupil who could not even effectively rebel!

I think the following question always arises at this junction: how soon do you take the child to be a fully responsible student, a real researcher? My answer is, from the very start. We can always present agendas to children, both in intellectual terms and in occupational terms. And if their assessments are erroneous, at least they are theirs. We learn from both ethology and developmental psychology that it is useless to work on a stage prior to the student's ability to be in that stage. And he will arrive quickest at that stage by proofs and refutations, where he can be aided by his teacher — so it is not the discovery method — where he can see where he is going — so he needs no phony motivation — and where his progress is the correction of his error. The Lakatos method has the merit of taking the student from where he stands and using his interruptions of the lecture as the chief vehicle of his progress, rather than worrying about the teacher's progress.

But there is more to it. We still introduce geometry using the axiomatic method — without knowing why, without being able to explain why. And often enough we use a variant of Euclid. This is wrong. The axioms only bewilder innocent students. We still teach the multiplication table as if there are no pocket computers. But children know there are and view standard arithmetic instruction as sheer punishment. I put all this as empirical findings.

There is still more to this. We all know of *idiot savants*. These are people able to do arithmetic like computers. They don't have to be idiots — for example, Von Neumann was an *idiot savant* — but it helps. Why? This is an intriguing question. We can try to find the answer in another and similar phenomenon, the infant musical prodigy. In the last century it was taken for granted that only a *Wunderkind* could become a concert pianist. Today we know otherwise. In the last century, however, all concert pianists had been *Wunderkinder*. Why? Because all the others had piano teachers who insisted on their keeping their elbows to their waists. In other words, *Wunderkinder* escaped instruction. Perhaps the same holds for *idiots savants*. If we assume that physically all brains are similar enough, we may assume all brains are pocket computers, and school usually destroys them. If so, we can help kids learn arithmetic without pressure and see whether every child can be made an *idiot* *savant*. After all, there is the Japanese school that by this method makes children into musical prodigies! We can play with them, counting up and down, and in all sorts of series, and see how well they learn the names of numbers, and of operations without learning them as operations; and if they get the results of the operations right we may soon have grandchildren who will not purchase pocket computers for want of any need for them.

This is frankly a speculation, and one that may well be refuted Still, I hope I am allowed to conclude with an empirical observation of how arithmetic is taught in vast portions of the western world. It is a fact easy to assess by seeing what teachers' training colleges advise their students to do and how this affects their conduct with children. The schoolma'am — the sexism is not mine but of the system — begins with counting concrete objects and adding concrete objects as a preparation for the abstraction from objects to numbers (I interrupt the story for a comment It is not clear whether concrete cases such as two apples are easier to comprehend than abstract cases such as the number two By Frege and Russell this is so; by Zermelo amd Fraenkel it is not; and as to Peano I cannot say. Staying within the system of cardinals, though, is the Frege-Russell view better than the Zermelo-Fraenkel? Both Fraenkel-bar-Hillel and Quine abstain from judgment.) But our schoolma'am does not know. She is convinced that the concrete counting is simpler and should be abstracted to abstract counting In this idea of abstraction she is very advanced and unknowingly follows Dedekind *Was sind und was sollen die Zahlen*? Except that no mathematician today agrees. Our ma'am is a bit apprehensive since she was told in a didactics class to prepare the ground well for the jump After many boring repetitions of two plus two this make four this, and two plus two that make four that, she asks the brightest kid in class what is two plus two. She is nervous and therefore so is he; he fumbles; she gets impatient; he gets lost; she takes a grip on herself and goes a step back She asks him: how many are two this plus two this? Four this, he mumbles. She encourages him. And two that plus two that? she triumphs. Four that, he answers, encouraged. And two plus two? He fumbles again

I interrupt again for a comment The mistake was to choose a bright kid; he smells a rat. He cannot articulate his trouble, but being troubled makes him an excellent Russellian, perhaps even a potential Lakatosian But the schoolma'am loses her nerve She knows the right moves, makes them, and fails; her self-confidence is shattered. She blames her pupil for treason and threatens to withhold her love He breaks down and says, four Yes, he knew the expected answer all along, but hated it. He is in the right and is told he is in the wrong His only way to maintain his independence is to say, I am bright but am no good at math; I will move to poetry. Except that he may have the same experience there too.

This little drama is no creation of my frustrated theatrical talent. I have seen this happen in schools in several countries, in arithmetic and in algebra and in geometry. I have no time to report the case of solving equations of the second order and analysing that case — I have done so in another paper of mine My point is, this ought to be stopped

I have thus arrived at the end of my first part, my critique. There remains the design of the new program and the way to effect the transition I will now conclude this presentation with one brief point on these two items. Any of you who has tenure and feels very brave and experimental can try the following recipe. First, let him treat his students as respectfully as possible and not motivate them, and explain frankly to them his conduct and position and situation as best he can. Second, let him discuss with them the shortcomings of the system as best he can. Third, let them discuss the design of the new system, and the possible ways of effecting the change, as equals with him as moderator. I cannot do so now I have a few papers and two books on the topic I cannot get them published. I was amazed and delighted with the fact that this group has invited me to express my very uncomplimentary view of the state of the art to which you are devoted: fully or in part, but committed to it; and I can only express my ardent wish to see the members here go home and start to implement the Lakatosian revolution Whether it goes this way or that way, it is bound to do good and be most exciting.

A REQUEST

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